



ELSEVIER

Journal of Geometry and Physics 39 (2001) 97–128

JOURNAL OF
GEOMETRY AND
PHYSICS

The symplectic and twistor geometry of the general isomonodromic deformation problem

N.M.J. Woodhouse

The Mathematical Institute, University of Oxford, 24-29 St. Giles, Oxford OX1 3LB, UK

Received 28 July 2000

Abstract

Hitchin's twistor treatment of Schlesinger's equations is extended to the general isomonodromic deformation problem. It is shown that a generic linear system of ordinary differential equations with gauge group $SL(n, \mathbb{C})$ on a Riemann surface X can be obtained by embedding X in a twistor space \mathcal{Z} on which $sl(n, \mathbb{C})$ acts. When a certain obstruction vanishes, the isomonodromic deformations are given by deforming X in \mathcal{Z} . This is related to a description of the deformations in terms of Hamiltonian flows on a symplectic manifold constructed from affine orbits in the dual Lie algebra of a loop group. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 32L25; 53C15

Subj. Class.: Spinors and twistors; Quantum field theory; Dynamical systems

Keywords: Symplectic geometry; Twistor geometry; Isomonodromic deformation

1. Introduction

The study of isomonodromic deformations of systems of ordinary differential equations in the complex plane was a significant topic at the beginning of the last century, when the classical work of Painlevé, Schlesinger, and Fuchs was published. It has come back into view in more recent years through connections with quantum field theory [3,16–18], differential geometry [6,7], and the theory of integrable systems (see, e.g. [1,14]).

In this paper, I shall explore in detail one aspect of the modern theory, suggested by Hitchin [5]. He considered the twistor space of a four-dimensional self-dual Riemannian manifold with $SU(2)$ symmetry. This is a three-dimensional complex manifold \mathcal{Z} , in which there is a four-dimensional family of projective lines corresponding to the points of the

E-mail address: nwood@maths.ox.ac.uk (N.M.J. Woodhouse).

0393-0440/01/\$ – see front matter © 2001 Elsevier Science B.V. All rights reserved.

PII: S0393-0440(01)00003-1

original manifold and on which the symmetry group acts holomorphically. The action is generated by three holomorphic vector fields which are independent in an open subset, but dependent on a special divisor S . By taking the vector fields as basis vectors, the tangent space to \mathcal{Z} is identified with $\mathfrak{sl}(2, \mathbb{C})$ at each point of the open subset. Thus the action determines a flat holomorphic connection on the trivial $\mathrm{SL}(2, \mathbb{C})$ bundle over the open orbit.

The restriction of the connection to a twistor line is an $\mathfrak{sl}(2, \mathbb{C})$ -valued meromorphic 1-form with poles at the intersections with S . In the case that Hitchin considered, there are four poles and the 1-form determines a Fuchsian system with four regular singularities. As the line is moved within the family, the poles move, but the monodromy of the system, which is the same as the holonomy of the flat connection, remains unchanged. By calling on the classical theory, therefore, one obtains from this geometrical picture a solution to the sixth Painlevé equation. Hitchin then goes on to exploit this correspondence to construct self-dual Einstein metrics from certain Painlevé transcendents.

Hitchin's correspondence between twistor manifolds with symmetry and isomonodromic families of ordinary differential equations holds more generally. In this paper, I shall follow through the details of his suggestion for the class of isomonodromic deformations considered by Jimbo et al. [9–11]. This enables one to understand their results within the framework of the general deformation theory of Kodaira [12].

In the general setting, we are given a complex Lie group G , a Riemann surface X , and a meromorphic 1-form α on X with values in the Lie algebra \mathfrak{g} . We pick a local coordinate z and write $\alpha = -A dz$. Then the equation $dy + \alpha y = 0$ becomes a system of linear ordinary differential equations

$$\frac{dy}{dz} = Ay, \tag{1}$$

where y is a fundamental solution, taking values in G , and A is a meromorphic function of z . The first question concerns the existence of twistor spaces: this is answered by Proposition 1, which gives the existence of an embedding of X in a complex manifold \mathcal{Z} on which \mathfrak{g} acts, with the generators independent on an open set, and from which α can be recovered by Hitchin's construction. This structure is not unique, however, even if we restrict attention to a small neighbourhood of X in \mathcal{Z} . If A has irregular singularities, then there are different choices for the way in which a divisor S can be attached to the open set so that the whole of X is embedded, including the singularities. Different choices give different possibilities for the normal bundle N of X . By Kodaira's theorem, the normal bundle determines the deformations of X in \mathcal{Z} : if

$$H^1(X, N) = 0 \quad \text{and} \quad \dim H^0(X, N) = d_0,$$

then X is one of d_0 -parameter family of compact curves X_t , on each of which the \mathfrak{g} -action gives a linear system of differential equations. These are isomonodromic (Proposition 6). It is shown that there is a natural choice for the twistor space (the 'full twistor space' in Definition 2), for which the parameter space has the largest possible dimension in the generic case; a full twistor space exists generically (Proposition 2), and is unique in a

neighbourhood of X (Proposition 4). In the full case, N can be constructed directly from α ; if $H^1(X, N) = 0$, as is the case if $X = \mathbb{C}\mathbb{P}_1$, and generally if α has enough singularities, then every isomonodromic deformation arises from this construction (Proposition 8).

A second theme of this paper is the Hamiltonian nature of the isomonodromic deformation equations. A *Fuchsian system* on $\mathbb{C}\mathbb{P}_1$ is a system with regular singularities of the form

$$\frac{dy}{dz} = \sum \frac{A_i y}{z - a_i},$$

where the residues $A_i \in \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ are independent of z . Apart from gauge and coordinate transformations, the only possible deformations in the generic case are given by moving the poles a_i . The monodromy is then preserved if and only if

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j}, \quad i \neq j, \quad \frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}$$

(Schlesinger’s equations).¹ Hitchin [8] interpreted these as a Hamiltonian flow on the coadjoint orbits of the A_i s. This also generalises: when irregular singularities are present, the flows are on symplectic manifolds constructed from affine orbits in the loop algebra. The symplectic forms can be written down explicitly in terms of α and the Stokes’ matrices, and, at least in the case $X = \mathbb{C}\mathbb{P}_1$, one can also find explicit expressions for the Hamiltonians (Proposition 9). The symplectic structure is related to the structure of \mathcal{Z} in a neighbourhood of S .

An appendix outlines the theory of isomonodromic deformations for linear systems on a general Riemann surface.

2. Twistor spaces

We suppose that we are given a complex Lie group G , a Riemann surface X , and a meromorphic 1-form α on X with values in the Lie algebra \mathfrak{g} . Our starting point is to interpret a solution $y : X \setminus \{\text{poles}\} \rightarrow G$ to the equation

$$dy + \alpha y = 0 \tag{2}$$

as a complex curve in G and to think of α as the pull-back of the Maurer–Cartan form on G —the \mathfrak{g} -valued 1-form whose contraction with a left-invariant vector field is the corresponding element of \mathfrak{g} .

In a local coordinate z on X , (2) is a linear system of ordinary differential equations of the form

$$\frac{dy}{dz} = Ay,$$

where A is meromorphic with values in \mathfrak{g} . Its poles are the *singularities* of the system— a pole of order $r + 1$ is a singularity of *rank* r . Some familiar results about such systems are summarised in the appendix.

¹ One pole is fixed at infinity, and has residue $-\sum A_i$.

Of course y is singular at the poles of α , and it is multi-valued, so the embedding in G is defined only locally. In the twistor picture, we seek to replace G by a complex manifold \mathcal{Z} of the same dimension on which the right action of G is retained and in which the singular points are included. In principle, the construction involves (i) taking a quotient by the monodromy group to make y single-valued and (ii) attaching hypersurfaces on which the right action of G action is not free. The intersections of X with the hypersurfaces will then correspond to the poles of the differential equation. Except for very special equations, however, the quotient is not Hausdorff. The best that we can do in general is to construct \mathcal{Z} as a neighbourhood of X , with the action of G replaced by an action of its Lie algebra (which is enough to determine α). The second step is generally straightforward in the regular case, but is more subtle when the linear system has irregular singularities.

In the context of the deformation problem, we shall adopt a special understanding of the meaning of the term ‘twistor space’.

Definition 1. A *twistor space* is a complex manifold \mathcal{Z} together with

1. a homomorphism from a complex Lie algebra \mathfrak{g} into the Lie algebra of holomorphic vector fields on \mathcal{Z} ; and
2. a smooth compact complex curve $X \subset \mathcal{Z}$

such that the induced linear map $\phi_z : \mathfrak{g} \rightarrow T_z\mathcal{Z}$ is an isomorphism for some $z \in X$.

Note that $\dim \mathcal{Z} = \dim \mathfrak{g}$. For the most part, we shall take $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, but other examples will also be considered.

Given a basis in \mathfrak{g} , $\Delta = \det \phi$ is a holomorphic section of $\wedge^{\dim \mathcal{Z}} T\mathcal{Z}$. We shall make the regularity assumptions that

$$S = \{\Delta = 0\}$$

is a complex hypersurface, that X is transversal to S , that S is the union of a finite set of components S_i , and that Δ has a zero of order $r_i + 1$ on S_i . These hold in all the twistor spaces constructed below.

At each $z \in \mathcal{Z} \setminus S$, define $\theta_z \in T_z^*\mathcal{Z}$ by $\theta_z = \phi_z^{-1}$. Then θ is a holomorphic 1-form on $\mathcal{Z} \setminus S$ with values in \mathfrak{g} . It is meromorphic on \mathcal{Z} and satisfies the Maurer–Cartan equation

$$d\theta + \theta \wedge \theta = 0. \tag{3}$$

Equivalently, $d + \theta$ is a flat meromorphic connection on a trivial bundle over \mathcal{Z} . The restriction $\alpha = \theta|_X$ determines a system of the form (2), with poles at the intersection points with S . We then say that (\mathcal{Z}, X) is a twistor space for the system.

Example. Let $G = \mathrm{SL}(n, \mathbb{C})$ and let $\mathfrak{t} \subset \mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ denote the diagonal subalgebra. As an $(n - 1)$ -dimensional additive group, \mathfrak{t} acts on itself by translation, and the action extends to the compactification $\mathbb{C}\mathbb{P}_{n-1}$ when we add a hyperplane at infinity. We also have the left action of \mathfrak{t} on G , defined by

$$g \mapsto \exp(A)g, \quad A \in \mathfrak{t}.$$

We put

$$\mathcal{Z} = G \times \mathbb{C}\mathbb{P}_{n-1}/\mathfrak{t}.$$

Then the right action of G on the first factor descends to the quotient.

We can think of \mathcal{Z} as being formed by attaching a single hypersurface S to G (the projection of the hyperplane at infinity in $\mathbb{C}\mathbb{P}_{n-1}$). The effect is to compactify the one-parameter subgroups generated by the semisimple elements of \mathfrak{g} . If $A \in \mathfrak{t}$ generates a closed subgroup, then $\{tA\} \subset \mathfrak{t}$ compactifies to a projective line in $\mathbb{C}\mathbb{P}_{n-1}$, and this in turn projects onto an embedded copy of $\mathbb{C}\mathbb{P}_1$ in \mathcal{Z} . The corresponding system of linear equations is

$$\frac{dy}{dz} = Ay,$$

which has a singularity of rank 1, a double pole, at infinity (the intersection with S).

Example. Suppose that X has genus g . Let $G = \mathcal{Z}$ be the Jacobi variety (an abelian group with Lie algebra $\mathfrak{g} = \mathbb{C}^g$) and let $X \hookrightarrow \mathcal{Z}$ be the standard embedding (see, e.g. [4, p. 87]). The corresponding system is

$$\frac{dy}{dz} = A,$$

where $A = (\xi_1, \dots, \xi_g)$ with the ξ s a basis for the holomorphic differentials on X . Here there are no singularities.

2.1. Existence of twistor spaces

Does every system of ODEs of the form (2) have a twistor space? Since the restriction of θ to X cannot vanish, a necessary condition is that $\alpha \neq 0$ at every point of X . This condition holds in the generic case (since it fails only if the all the entries in the matrix A have a coincident zero). It is also sufficient.

Proposition 1. *Let α be a meromorphic \mathfrak{g} -valued 1-form on X with no zeros. Then the linear system of ODEs $dy + \alpha y = 0$ has a twistor space.*

Proof. We construct \mathcal{Z} by taking a quotient of a neighbourhood of the identity section X in $X \times G$ by a distribution F constructed from the linear system.

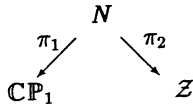
Let D be a neighbourhood of a pole a not containing any of the other poles, and let z be a coordinate on D such that $z = 0$ at a . Then $\alpha = -A dz$ in D , where $A : D \setminus \{0\} \rightarrow \mathfrak{g}$ is holomorphic and has a pole of order $r + 1$ at $z = 0$.

Define F to be the distribution on $D \times G$ tangent to the non-vanishing vector field

$$z^{r+1}(\partial_z - R_A),$$

where R_A is the right-invariant vector field on G generated by $A(z)$. If D' is an open set not containing any other poles, then we define F in the same way on $D' \times G$, but without the

factor z^{r+1} ; i.e., F is tangent to $\partial_z - R_A$. The vector fields are proportional on $D \cap D'$, so F is well defined globally as a distribution on $X \times G$. Under the condition on α , we have $F \cap T_x X = 0$ at every $x \in X$. So it is possible to choose an open neighbourhood N of X in $X \times G$ such that the quotient $\mathcal{Z} = N/F$ is a Hausdorff complex manifold of the same dimension as G . We then have a double fibration



and a smooth curve $\pi_2(X) \subset \mathcal{Z}$, which we also denote by X .

Because we are looking only at a neighbourhood of the identity section, the right action of G on $X \times G$ does not pass to \mathcal{Z} ; but the corresponding Lie algebra action does. Each $v \in \mathfrak{g}$ can be identified with a left-invariant vector field on G , and hence with a vector field on $X \times G$ tangent to the fibres of π_1 . Its projection V by π_{2*} is a holomorphic vector field on \mathcal{Z} , and the map $v \mapsto V$ is a Lie algebra representation, satisfying the conditions in the definition of a twistor space. The singular hypersurface has components given by the poles of α , and X meets these transversally.

It remains to show that $\theta|_X = \alpha$. To do this, we note that the meromorphic vector field $\partial_z - R_A$ on $\mathbb{C}P_1 \times G$ is tangent to F , and so its projection into \mathcal{Z} vanishes. On the other hand, at the identity, the right- and left-invariant vector fields generated by an element of \mathfrak{g} coincide. Hence $i_{\partial_z} \theta = -A$. The proposition follows. \square

Remark. If we instead take the curve in \mathcal{Z} to be the projection under π_2 of $X \times \{g\}$ for some other constant element of g , then we obtain instead a twistor space for $g^{-1}\alpha g$.

In the irregular case, the twistor space is not unique: the one that arises in Proposition 1 is *minimal* in a sense that will be explained later.

2.2. Full twistor spaces

The difference in structure between different twistor spaces of a system of ODEs can be understood by looking at the structure in a neighbourhood of a point of $a \in S$. By introducing a local coordinate z that vanishes on S , we can choose a neighbourhood U of the form $S \times D$, where $D \subset \mathbb{C}$ is, say, the unit disc, and $X \cap U$ is $\{(a, z)\}$, $z \in D$.

Suppose that $\alpha = -A dz$ has a pole of order $r + 1$ at $z = 0$. Then corresponding system

$$\frac{dy}{dz} = Ay \tag{4}$$

has a singularity of rank r at $z = 0$.

Given a holomorphic vector field V on \mathcal{Z} tangent to S , we can construct a holomorphic family of copies D_t of D in U by moving $D_0 = X \cap U$ along V (and if necessary restricting to a smaller neighbourhood of a): here t is a complex parameter taking values in some neighbourhood of $t = 0$. By restricting θ to each X_t , we get a one-parameter family of

ODEs

$$\frac{dy}{dz} = A(z, t)y,$$

each with a singularity of rank r at $z = 0$ (the singularity does not move with t because V is tangent to S). It follows from the flatness of $d + \theta$ that

$$\frac{\partial A}{\partial t} = \frac{\partial \Omega}{\partial z} - [A, \Omega],$$

where $\Omega = -i_V \theta$. This is the *local deformation equation* (see appendix).

At each fixed value of t , Ω is a function of the coordinate z . Introducing the notation

$$\nabla \Omega = \partial_z \Omega - [A, \Omega],$$

we have

$$\Omega = O(z^{-r-1}), \quad \nabla \Omega = O(z^{-r-1}) \quad \text{as } z \rightarrow 0. \tag{5}$$

When the singularity is irregular, the various twistor spaces differ in the extent to which the converse holds: in the minimal construction, an Ω satisfying these conditions is of the form $i_V \theta$ for some holomorphic V only if $\Omega - f(z)A$ is holomorphic at $z = 0$ for some holomorphic function f .

Definition 2. A twistor space \mathcal{Z} is full at $a \in S$ if for every $\Omega : D \rightarrow \mathfrak{g}$ such that (5) holds, there is a holomorphic vector field V on $U \subset \mathcal{Z}$ such that $\Omega = -i_V \theta|_X$. The twistor space is full if it is full at every point of S .

When the system has irregular singularities, and $\text{Rank}(G) > 2$, the twistor space constructed in Proposition 1 is not full. We can see this by noting that, for any holomorphic V , $i_V \theta$ has singular part at each pole that is proportional (by a holomorphic function) to a multiple of A , and cannot, therefore, give rise to the most general Ω satisfying (5). We shall put this more precisely below when we consider the normal bundle of X in \mathcal{Z} .

A full twistor space generates not only the ODE itself, but also its isomonodromic deformations. We shall see that it is possible to construct a full twistor space in the generic case, but there are some rather special exceptions. A necessary condition for existence is that if Ω and Ω' both satisfy (5), and if

$$\Omega = \frac{M}{z^{r+1}} + O(z^{-r}), \quad \Omega' = \frac{M'}{z^{r+1}} + O(z^{-r})$$

as $z \rightarrow 0$, with $k > 0$, then $[M, M'] = 0$. This fails in the following class of examples.

Example. Suppose that

$$A = z^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\Omega = \frac{M}{z}, \quad \Omega' = \frac{M'}{z},$$

satisfy (5) for any constant M, M' ; but both cannot be generated from vector fields in the same twistor space if $[M, M'] \neq 0$.

2.3. The singular hypersurface

We now look in detail at the structure of a twistor space in a neighbourhood of S . The fullness condition at a point of $X \cap S$ is this: given Ω such that (5) holds, is $\phi(\Omega)$ holomorphic at $z = 0$? Let us write

$$A = \frac{p}{z^{r+1}} + H, \quad \Omega = \frac{\omega}{z^{r+1}}, \quad (6)$$

where

$$p = p_0 + p_1 z + \cdots + p_r z^r, \quad \omega = \sum_0^{\infty} \omega_i z^i,$$

and H is holomorphic on U . If $\omega = O(z^{r+1})$ as $z \rightarrow 0$, then Ω is holomorphic at $z = 0$ and can be generated by the holomorphic vector field $\phi(\Omega)$ in any twistor space for A ; so the source of any difficulty lies in the first r terms in the Taylor expansion of ω .

When we separate out the coefficients of $z^{-2(r+1)}, z^{-2r-1}, \dots, z^{-(r+2)}$ in (5), we obtain for $r > 0$

$$\begin{aligned} [p_0, \omega_0] &= 0, \\ [p_0, \omega_1] + [p_1, \omega_0] &= 0, \dots, [p_0, \omega_{r-1}] + [p_1, \omega_{r-2}] + \cdots + [p_{r-1}, \omega_0] = 0, \\ [p_0, \omega_r] + [p_1, \omega_{r-1}] + \cdots + [p_r, \omega_0] - (r+1)\omega_0 &= 0, \end{aligned}$$

or in the case $r = 0$,

$$[p_0, \omega_0] = \omega_0. \quad (7)$$

The generic case is that one or other of the following hold:

1. $r > 0$ and the eigenvalues of p_0 are distinct; or
2. $r = 0$, the eigenvalues of p_0 are distinct, and no pair differ by an integer.²

Lemma 1. *If A is generic, then Ω satisfies (5) if and only if $\omega_0 = 0$ and $[\omega, p] = O(z^{r+1})$ as $z \rightarrow 0$.*

Proof. Under either of the conditions 1, 2, the eigenvalues of p can be assumed to be distinct, since they are distinct at $z = 0$ and since we can, if necessary, replace U by a smaller neighbourhood. So we can find a holomorphic gauge transformation $g : U \rightarrow G$ such that $g^{-1} p g$ is the sum of a diagonal polynomial and a term that vanishes to order z^{r+1} at $z = 0$, and so can be absorbed into r .

² To prove Lemma 1 and Proposition 2, we only need that no pair should differ by 1; however, the stronger condition here is needed to construct g_f (see appendix), and is imposed here to avoid special cases in the presentation below.

If we assume first that p is actually diagonal, then we deduce successively that $\omega_0, \dots, \omega_{r-1}$ are diagonal (for $r > 0$) and that

$$[p_0, \omega_r] = (r + 1)\omega_0.$$

For $r > 0$, this implies that $\omega_0 = 0$ since the diagonal terms on the left-hand side vanish, and hence that ω_r is also diagonal. For $r = 0$, it gives $\omega_0 = 0$ since p_0 has no pair of eigenvalues differing by 1. Thus, whether or not p is diagonal, we have that Ω is holomorphic at $z = 0$ when $r = 0$; and that when $r > 0$,

$$\Omega = \frac{gqg^{-1}}{z^r} + O(z^0),$$

where q is a diagonal polynomial of degree $r - 1$. □

Proposition 2. *Let $dy + \alpha y = 0$ be a generic system with $G = \text{SL}(n, \mathbb{C})$. Then there exists a full twistor space.*

Proof. Any twistor space is full at a singularity of rank $r = 0$ since any Ω satisfying (5) is then holomorphic at $z = 0$, and can, therefore, be generated by a holomorphic vector field in any twistor space. In the irregular case, we construct \mathcal{Z} from the ‘minimal’ twistor space in Proposition 1, by cutting out and replacing a neighbourhood of each component of S corresponding to an irregular singularity.

Suppose, to begin with, that the system has a singularity of rank r at $z = 0$ and that in a neighbourhood D of $z = 0$ we have $\alpha = -A dz$, where

$$A = \frac{p}{z^{r+1}} + H \tag{8}$$

with p a diagonal polynomial of degree r with distinct diagonal entries throughout D and H holomorphic. By making a diagonal gauge transformation, we can make H off-diagonal.

Pick constant diagonal matrices q_1, \dots, q_{n-2} which, together with $p(0)$, form a basis for the diagonal subalgebra of \mathfrak{g} , and for each i let H_i be the off-diagonal matrix with entries

$$(H_i)_{ab} = \frac{zH_{ab}(q_{ia} - q_{ib})}{p_a - p_b}, \quad a \neq b,$$

where q_{ia} and p_a , $a = 1, \dots, n$, are the diagonal entries in q_i and p . Thus

$$[p, H_i] = [q_i, zH], \quad [q_i, H_j] = [q_j, H_i]. \tag{9}$$

Now introduce evolution equations for the diagonal matrix $p(z)$ and the off-diagonal matrix $H(z)$ as functions of the complex variables t_1, \dots, t_{n-1} by putting

$$\partial_i p = -rq_i, \quad \partial_i H = \partial_z H_i - [H, H_i],$$

where $\partial_i = \partial/\partial t_i$, $\partial_z = \partial/\partial z$. The integrability of this system is established by showing that

$$\partial_i H_j - \partial_j H_i = [H_i, H_j].$$

Since both sides are off-diagonal, this follows from (9) and

$$\begin{aligned} [p, \partial_i H_j] - [p, \partial_j H_i] &= z[q_j, \partial_i H] - z[q_i, \partial_j H] \\ &= z[q_j, \partial_z H_i] - z[q_i, \partial_z H_j] - z[q_j, [H, H_i]] + z[q_i, [H, H_j]] \\ &= z[H_i, [q_j, H]] - z[H_j, [q_i, H]] = [p, [H_i, H_j]]. \end{aligned}$$

So the evolution equations extend H and p to functions of (z, t_1, \dots, t_{n-2}) on a neighbourhood W of the origin in \mathbb{C}^{n-2} .

It follows from the definitions that

$$\nabla = d - \frac{p dz}{z^{r+1}} - H dz - \sum_i \left(\frac{q_i dt_i}{z^r} + H_i dt_i \right)$$

is flat meromorphic connection on the trivial bundle principal bundle $P = G \times W$.

Let \mathcal{Q} denote the quotient of a neighbourhood of the identity section in P by the horizontal foliation. The foliation extends holomorphically to $z = 0$ since it is spanned by

$$z^{r+1} \partial_z - p - z^{r+1} H, \quad z^r \partial_i - q_i - z^r H_i \quad (i = 1, \dots, n-2),$$

where p, q_i, H, H_i are interpreted as right-invariant vector fields on G .

The quotient is a ‘local twistor space’ for A in the sense that it carries a holomorphic \mathfrak{g} -action, which is free and transitive except on the hypersurface $S' = \{z = 0\}$, and contains a copy of D on which the induced system is α . Moreover, the fullness condition holds at S since p_0 and the q_i s span the diagonal subalgebra (in the case $n = 2$, p_0 on its own does that). Any generic A can be reduced to the form (8) by a holomorphic gauge transformation $g(z)$; so more generally a local twistor space can be constructed by applying the same gauge transformation to ∇ .

By using the \mathfrak{g} action, we can identify $\mathcal{Q} \setminus S'$ with $V \setminus S$, where V is a neighbourhood in \mathcal{Z} of the $z = 0$ intersection point of S and X . Then the embedded copy of $D \setminus \{0\}$ is mapped onto a punctured neighbourhood of the singularity in X . The identification allows us to replace V by \mathcal{Q} . By repeating this for the other irregular singularities, we obtain a full twistor space. \square

Given the choice of coordinate z in a neighbourhood of an irregular singularity, p_0 is a well-defined map $S \rightarrow \mathfrak{g}$. Up to scale, p_0 is independent of the choice of z . We thus have a natural map $[p_0] : S \rightarrow \mathbb{P}\mathfrak{g}$. It is equivariant with respect to the \mathfrak{g} action on S and the adjoint action on $\mathbb{P}\mathfrak{g}$. The following is immediate from the proof above.

Proposition 3. *A twistor space for a generic system is full at an irregular singularity if and only if $[p_0] : S \rightarrow \mathbb{P}\mathfrak{g}$ is regular at $X \cap S$.*

When the space is not full, $[p_0](S)$ has non-zero codimension in $\mathbb{P}(\mathfrak{g})$. In the generic case, the full twistor space is locally unique in the sense of the following proposition.

Proposition 4. *Suppose that (\mathcal{Z}, X) and (\mathcal{Z}', X) are full twistor spaces for a generic linear system of ODEs on a Riemann surface X with $G = \mathrm{SL}(n, \mathbb{C})$. Then there are neighbourhoods*

$U \supset X$ and $U' \supset X$ in \mathcal{Z} and \mathcal{Z}' and a \mathfrak{g} -equivariant biholomorphic map $\rho : U \rightarrow U'$ such that $\rho(X) = X$.

If we exclude the poles from X and the corresponding hypersurfaces S and S' from \mathcal{Z} and \mathcal{Z}' , then ρ is determined in a straightforward way by the \mathfrak{g} actions on \mathcal{Z} and \mathcal{Z}' . It is defined by choosing a (multi-valued) solution y to the system on X and then extending y to a (multi-valued) map from a neighbourhood of $X \setminus S$ in $\mathcal{Z} \setminus S$ to G such that $dy + \theta y = 0$. Similarly, y extends to y' on \mathcal{Z}' . Then the required map is $\rho = y'^{-1} \circ y$, where the branches are chosen so that ρ is the identity on X (ρ is well defined since y and y' have the same holonomy). The fact that ρ extends holomorphically to S in the full case is a corollary of Proposition 6 below.

2.4. The structure of S

Suppose that $G = \text{SL}(n, \mathbb{C})$ and that (\mathcal{Z}, X) is full. We denote by Γ the space of parametrised curves

$$D \rightarrow \mathcal{Z} : z \mapsto \gamma(z),$$

where $D \subset \mathbb{C}$ is the unit disc, $\gamma(D)$ meets some component of S transversally at $z = 0$, and γ extends smoothly to $|z| = 1$.

We shall now develop a picture in which $\theta_\gamma = \gamma^*(\theta)$ is regarded as an element of the dual of Lie algebra of the loop group LG .³ Different elements of Γ give different points of an orbit in $L\mathfrak{g}^*$ of an affine action of LG . We shall construct a finite-dimensional complex symplectic manifold from the orbit which determines the singular part of α at $z = 0$.

2.5. The Fuchsian case

If the singularity at $z = 0$ is regular and generic, then any twistor space is full at $z = 0$ and any holomorphic map $z \rightarrow \Omega(z) \in \mathfrak{g}$ generates a local holomorphic vector field tangent Y to S . Moreover, we can write

$$\theta = p_0 d(\log z) + \theta',$$

where $z = 0$ on S and θ' is holomorphic on S .⁴ Then p_0 is independent of the choice of the function z , and therefore determines a natural map $\mu : S \rightarrow \mathfrak{g} = \mathfrak{g}^*$ — the identification being given by the bilinear form $\text{tr}(\xi_1, \xi_2)$, $\xi_i \in \mathfrak{g}$. The image $\mu(S)$ is open subset of a coadjoint orbit.

By evaluating Ω and Y at $z = 0$, we obtain a natural identification

$$T_a S = \mathfrak{g}/[p_0(a)], \quad a \in S.$$

³ An element of loop algebra is a map $B : S^1 \rightarrow \mathfrak{g}$. We define $\langle \theta_\gamma, B \rangle$ by integrating $\text{tr}(B\theta_\gamma)$ around the unit circle.

⁴ Thus θ has a logarithmic pole in the sense of Malgrange [13].

We can, therefore, define a 2-form σ on S by

$$\sigma_a(Y, Y') = \text{tr}(p_0[\Omega, \Omega']).$$

This is closed and presymplectic since it is the pull back to S by μ of the symplectic form on $\mu(S)$.

2.6. The irregular case

In the irregular case, the analogous structure involves information from the higher formal neighbourhoods of S . It arises from the action on Γ of the group L_+G of holomorphic maps $g : D \rightarrow G$ that extend smoothly to \bar{D} : if $g \in L_+G$ then $\gamma \in \Gamma$, then $(g\gamma)(z) = \gamma(z)g(z)$.⁵

A tangent vector Y to Γ at $[\gamma]$ is a section of $T\mathcal{Z}|_\gamma$, tangent to S at $z = 0$. Put

$$\sigma_\gamma(Y, Y') = \frac{1}{2\pi i} \oint \text{tr}(\Omega \nabla \Omega'), \quad (10)$$

where $\Omega = i_Y\theta$, $\Omega' = i_{Y'}\theta$, $\nabla\Omega = d\Omega + [\theta, \Omega]$ and the integral is along a loop surrounding $z = 0$. This form is closed, but degenerate (its closure follows from the construction below). Its characteristic distribution is integrable, by closure, and contains the Y s for which $\Omega = O(z^{r+1})$ as $z \rightarrow 0$. These span the orbits of the normal subgroup $L_{r+1}G \subset L_+G$ of maps $g : D \rightarrow G$ such that $g = 1 + O(z^{r+1})$ as $z \rightarrow 0$. Since $\Gamma/L_{r+1}G$ is finite-dimensional, the quotient Γ_r of Γ by the characteristic distribution is a finite-dimensional symplectic manifold.

Since \mathcal{Z} is full, the tangent space to Γ_r at $[\gamma]$ is the set of holomorphic maps

$$\Omega : D \setminus \{0\} \rightarrow \mathfrak{g},$$

such that (5) holds, modulo maps with zeros of order r at $z = 0$.

2.7. Affine orbits

Let LG denote the loop group of smooth maps $S^1 \rightarrow G$ [15]. Its Lie algebra is the space $L\mathfrak{g}$ of smooth maps $\Omega : S^1 \rightarrow \mathfrak{g}$.

A 1-form α on S^1 with values in \mathfrak{g} determines an element of $L\mathfrak{g}^*$ by

$$\langle \alpha, \Omega \rangle = \frac{1}{2\pi i} \oint \text{tr}(\Omega \alpha).$$

Let $\mathcal{A}_r \subset L\mathfrak{g}^*$ denote the subspace of smooth α s that extend meromorphically to D with a pole at the origin of order at most $r + 1$, and no other poles. Thus

$$L\mathfrak{g}^* \supset \mathcal{A}_r \supset \mathcal{A}_{r-1} \supset \cdots \supset \mathcal{A}_0.$$

Like any dual Lie algebra $(L\mathfrak{g})^*$ carries the standard Kostant–Kirillov–Souriau Poisson structure, which is preserved by the coadjoint action of LG . The natural symplectic arena

⁵ Of course this is well-defined only for g close to the identity; what follows has to be qualified in a similar way.

for the isomonodromy problem is not, however, that of the corresponding coadjoint orbits, but rather that of the orbits of the *affine action* of $L\mathfrak{g}$ on $L\mathfrak{g}^*$ given by the cocycle

$$c(\Omega, \Omega') = \frac{1}{2\pi i} \oint \text{tr}(\Omega \, d\Omega').$$

Each $\Omega \in L\mathfrak{g}$ determines a vector field on $L\mathfrak{g}^*$, its value at $A \in L\mathfrak{g}^*$ being given by

$$\langle \delta\alpha, \cdot \rangle = \langle \alpha, [\Omega, \cdot] \rangle - c(\Omega, \cdot).$$

The first term on the right-hand side is the usual coadjoint action; the second is a translation introduced by Souriau [19] (see also [20]).⁶ By integrating the flow on $L\mathfrak{g}^*$, we obtain the gauge action of LG :

$$\alpha \mapsto g^{-1}\alpha g + g^{-1} dg.$$

The symplectic structure on the corresponding orbits is⁷

$$\sigma(Y, Y') = c(\Omega, \Omega') - \langle \alpha, [\Omega, \Omega'] \rangle = \frac{1}{2\pi i} \oint \text{tr}(\Omega \nabla \Omega') = \frac{1}{2\pi i} \oint \text{tr}(\Omega \delta' \alpha), \quad (11)$$

where Y, Y' are the vector fields generating the actions of $\Omega, \Omega' \in L\mathfrak{g}$,

$$\nabla \Omega = d\Omega + [\alpha, \Omega],$$

and $\delta' \alpha$ is the variation induced by Y' . The flow of $\Omega \in L\mathfrak{g}$ is generated by

$$h(\alpha) = \frac{1}{2\pi i} \oint \text{tr}(\alpha \Omega).$$

However, $[h_A, h_B] = h_{[A, B]} + c(A, B)$, so the action is not Hamiltonian: the cocycle is the obstruction to the existence of a moment map.

2.8. The symplectic structure of M_r

Let \mathcal{O} be the affine orbit of some generic element of \mathcal{A}_r ; i.e., an element of the form

$$g^{-1} dg - g^{-1} \left(\frac{p}{z^{r+1}} + H \right) g,$$

where $g \in L_+G$, $H \in L_+\mathfrak{g}$ are holomorphic on D , and p is a polynomial in z with distinct eigenvalues for $z \in D$.

⁶ For a general Lie group, the affine action is symplectic, but not Hamiltonian when c is not a coboundary. In fact the affine orbits are the models for the non-Hamiltonian transitive symplectic actions of the group in the same way that the coadjoint orbits are the models for the Hamiltonian actions. If c is the obstruction to the existence of a moment map for some transitive symplectic action of the Lie algebra on a symplectic manifold M , then M can be mapped equivariantly and symplectically to an affine orbit in the dual Lie algebra. The affine orbits have an alternative description in terms of the coadjoint orbits of the central extension determined by c , but this is less convenient for our purposes here, see [15, p. 44].

⁷ This is a good definition of σ since the right-hand side vanishes whenever Ω_1 fixes A , and therefore whenever Y_1 vanishes at A . We shall not need to consider the precise sense in which σ is non-degenerate, since we shall deal only with finite-dimensional submanifolds.

A general element $\alpha \in \mathcal{O}$ is a smooth 1-form on S^1 with values in \mathfrak{g} . For any $\alpha, \alpha' \in \mathcal{O}$, the two systems

$$dy + \alpha y = 0, \quad dy + \alpha' y = 0$$

on the circle have the same monodromy matrix M up to conjugacy, since that is the condition that y and \hat{y} can be chosen so that $g = y(z)\hat{y}(z)^{-1}$ is single-valued; g is then the element of LG that maps one system into the other.

Since c vanishes on $L_{+}\mathfrak{g}$, the action of LG on \mathcal{O} is Hamiltonian when restricted to $L_{r+1}G$, the subgroup of loops of the form $1 + z^{r+1}h$, where h is holomorphic. We denote by \mathcal{M}_r the Marsden–Weinstein reduction of $\mu_{r+1}^{-1}(0)$, where μ_{r+1} is the moment map. That is, \mathcal{M}_r is the quotient of $\mathcal{A}_r \cap \mathcal{O}(A)$ by the action of $L_{r+1}G$. It is a finite-dimensional complex symplectic manifold, with symplectic form that we shall again denote by σ . Its points are elements of $\mathcal{A}_r \cap \mathcal{O}(A)$, modulo gauge transformations of the form

$$\alpha \mapsto g^{-1}dg - g^{-1}\alpha g, \quad g \in L_{r+1}G.$$

A tangent vector $Y \in T_{\alpha}\mathcal{M}_r$ is of the form

$$\delta\alpha = \nabla\Omega = d\Omega + [\alpha, \Omega],$$

where Ω satisfies (5); two such Ω s define the same tangent vector whenever their difference is in $L_{r+1}\mathfrak{g}$.

We can construct from $\alpha \in \mathcal{A}_r$ the following objects (see appendix).

1. The singularity data (m, t) , where t is a diagonal polynomial of degree $r - 1$ and m is the exponent of formal monodromy.
2. The formal series $g_f = \sum g_i z^i$.
3. The matrices C_i , defined as follows. For each α , choose a solution y to the corresponding system with fixed monodromy matrix M and choose $2r$ sectors \mathcal{S}_i at $z = 0$, as in (A.3). Then put $C_i = y_i(z)^{-1}y(z)$, where y_i is the corresponding special solution and we continue y in the positive sense around $z = 0$ into the sector \mathcal{S}_i . If we put

$$y_{2r+1} = y_1 e^{2\pi im}, \quad \mathcal{S}_{2r+1} = \mathcal{S}_1,$$

then $C_{2r+1} = e^{-2\pi im} C_1 M$ and we can define the Stokes' matrices by $S_i = C_i C_{i+1}^{-1}$ ($i = 1, \dots, 2r$).

These are not quite uniquely determined: we are free to make the replacement

$$g_f \mapsto g_f T, \quad C_i \mapsto T^{-1} C_i, \quad S_i \mapsto T^{-1} S_i T, \tag{12}$$

where T is diagonal and independent of z . We shall express the symplectic form on \mathcal{M}_r in terms of these variables.

Given t and the monodromy matrix M , the Stokes' matrices and the exponent of formal monodromy satisfy two constraints.

(C1) Let P_i be the matrix of the permutation that puts the real parts of diagonal entries in $z^{-r}t$ in increasing order in $\mathcal{S}_i \cap \mathcal{S}_{i+1}$. Then for each i , $P_i^{-1} S_i P_i$ is upper triangular and

$P_i S_{i+1} P_i^{-1}$ is lower triangular, both with ones on the diagonal. This follows from the fact that

$$\exp(z^{-r} t + m \log z) S_i \exp(-z^{-r} t - m \log z) \rightarrow 1 \tag{13}$$

faster than any power of z as $z \rightarrow 0$ in $\mathcal{S}_i \cap \mathcal{S}_{i+1}$.

(C2) The product $e^{-2\pi i m} S_1 \cdots S_{2r}$ is conjugate to M^{-1} .

Denote by \mathcal{C}_r the set matrices $S_i \in \text{SL}(n, \mathbb{C})$, m diagonal and trace-free, satisfying these two constraints. Given a point \mathcal{C}_r , we choose C_1 such that

$$e^{-2\pi i m} S_1 \cdots S_{2r} = C_1 M^{-1} C_1^{-1},$$

and define C_2, \dots, C_{2r+1} by $C_{i+1} = S_i^{-1} C_i$. Put

$$\omega = \frac{1}{2\pi i} \sum_1^{2r} \text{tr}(dC_i C_i^{-1} \otimes dS_i S_i^{-1}) + \pi i \text{tr}(dm \otimes dm) - \text{tr}(dC_1 C_1^{-1} \otimes dm). \tag{14}$$

It is shown in the appendix that ω is skew-symmetric, and in fact a symplectic form on \mathcal{C}_r .

For each point of \mathcal{M}_r , we pick a representative in $\alpha \in \mathcal{A}_r$. We then define 1-forms on \mathcal{M}_r by

$$\Theta = dg g^{-1} + \frac{g dt g^{-1}}{z^r}, \quad \gamma = g_0^{-1} dg_0,$$

where d is the exterior derivative on \mathcal{M}_r and g is the polynomial obtained by truncating the formal power series g_t at some large power of r . With this notation, the symplectic form on \mathcal{M}_r is given by the following proposition.

Proposition 5. *The symplectic form on \mathcal{M}_r is*

$$\sigma = \frac{1}{2\pi i} \oint \text{tr}(\Theta \wedge \nabla \Theta') + \text{tr}(\gamma \wedge dm) - \omega,$$

where $\nabla \Omega = \partial_z \Omega dz + [\alpha, \Omega]$.

The proof is by splitting the integral in the definition of σ into sections lying in the various sectors, and then shrinking the contour to zero. The details are given in the appendix.

The formula for σ is independent of the choices made in defining the variables on \mathcal{M}_r . In particular, it depends on the first r terms in the formal series since the right-hand side of the formula is unchanged when Θ is replaced by $dh h^{-1} + h \Theta h^{-1}$, where $h = 1 + O(z^{r+1})$. It is also unchanged when g, C_i are replaced by $gT, T^{-1} C_i$, where T is diagonal and independent of z .

We can see the local structure of \mathcal{M}_r from the proposition. The submanifolds on which g and t are constant (up to the freedom 12) are symplectomorphic to \mathcal{C}_r . While, those on which m and S_i are constant (up to 12) are symplectomorphic to a fixed manifold \mathcal{P}_r ; by mapping $[\gamma] \in \Gamma_r$ to $[\alpha]$, where $\alpha = \gamma^*(\theta)$, and by noting the coincidence of the formulas for the symplectic forms, we identify \mathcal{P}_r with Γ_r .

3. Local uniqueness of the full twistor space

Let \mathcal{P} denote the subset of $\mathcal{O} \cap \mathcal{A}_r$ given by fixing the values of the Stokes' matrices and exponent of formal monodromy. By using the actions of $L_{r+1}G$ on Γ and \mathcal{P} to pick representatives in $[\gamma]$ and $[\alpha]$, we can identify Γ with an open neighbourhood in \mathcal{P} so that $\gamma \in \Gamma$ corresponds to $\alpha \in \mathcal{P}$ such that $\alpha = \gamma^*(\theta)$. We then deduce the following proposition.

Proposition 6. *Suppose that (\mathcal{Z}, X) and (\mathcal{Z}', X) are full twistor spaces for a generic linear system of ODEs on a Riemann surface X with $G = \mathrm{SL}(n, \mathbb{C})$. Let $a \in X$ be a pole of order $r + 1 > 1$. Then there are neighbourhoods U, U' of a in \mathcal{Z} and \mathcal{Z}' and a \mathfrak{g} -equivariant biholomorphic map $\rho : U \rightarrow U'$ such that $\rho(a) = a$ and $\rho(X \cap U) = X \cap U'$.*

Proof. Choose a coordinate z on a small disc in X such that the pole is at $z = 0$, and extend this to a neighbourhood of a in \mathcal{Z} so that S is given by $z = 0$. Then we can identify a neighbourhood of a with $S \times D$, as before. For each $s \in S$, we have a holomorphic map $\gamma_s : D \rightarrow \mathcal{Z}$ and hence an element α_s of \mathcal{P} such that $\gamma_s^* = \alpha_s$. Let $\gamma'_s : D \rightarrow \mathcal{Z}'$ be the corresponding map into the second twistor space. Then the required biholomorphic map is $\rho : (s, z) \mapsto \gamma'_s(z)$. \square

Proposition 4 above is an immediate corollary, since Proposition 6 implies that the map ρ constructed there extends to S .

4. Isomonodromic deformations

We have shown that a generic $\mathrm{SL}(n, \mathbb{C})$ system of the form (2) on a Riemann surface can be generated from a twistor space (\mathcal{Z}, X) , and that, if we require \mathcal{Z} to be full, then it is unique, at least in a neighbourhood of X . In the case of Fuchsian equations on \mathbb{CP}_1 , Hitchin [5] showed that the isomonodromic deformations of the system are given by the deformations of X in \mathcal{Z} (every twistor space being full in the Fuchsian case). This is also true more generally, as we shall now see.

By Kodaira's theorem, the deformations of a compact curve $X \subset \mathcal{Z}$ are determined by the properties of the normal bundle $N = T\mathcal{Z}|_X/TX$. Put

$$d_1 = \dim H^1(X, N) \quad \text{and} \quad d_0 = \dim H^0(X, N).$$

When $d_1 = 0$, the theorem implies that X is one of a complete d_0 -parameter holomorphic family of embedded curves. The tangent space to the parameter space at X is naturally identified with $H^0(X, N)$ [12].

For each curve X in the family, we have a meromorphic 1-form $\theta|_X$ and hence a system of differential equations of the form (2). As we vary the curve along a path X_t in the family, $t \in [0, 1]$, the tangent at t is an element of $H^0(X_t, N)$. This we represent by local sections Y_i of $T\mathcal{Z}|_{X_t}$, chosen to be tangent to S at the poles. Thus the Y_i s are uniquely determined up to the addition local tangent vector fields to X_t that vanish at $S \cap X_t$. Put $\Omega_i = i_{Y_i}\theta$,

$\alpha_t = \theta|_{X_t}$, and identify local neighbourhoods in the X_t s along Y_t . Then Ω_i is meromorphic, with a pole of order r at a singularity of rank r . By (3),

$$\partial_t \alpha_t = \nabla_{\alpha_t} \Omega_i = d\Omega_i + [\alpha_t, \Omega_i].$$

Moreover, on the overlap of their domains, $\Omega_i - \Omega_j = i_{T_{ij}} \alpha_t dz$ for some tangent vector T_{ij} to X , which must vanish at any poles in the overlap. By using the results in the appendix, we deduce the following proposition.

Proposition 7. *Let $G = \text{SL}(n, \mathbb{C})$, let (\mathcal{Z}, X) be a twistor space, and let X' be a deformation of X . Then the linear system of ordinary differential equations on X' is an isomonodromic deformation of the linear system on X .*

4.1. The minimal twistor space

In the minimal case, we can find the normal bundle of X as follows. For $x \in X$, put $L_\alpha(x) = \alpha(T_x X) \subset \mathfrak{g}$ when x is not a pole; and

$$L_\alpha(x) = z^{r+1} \alpha(T_x X),$$

when x is a pole of rank r and $z = 0$ at x . Then $L_A \rightarrow X_0$ is a holomorphic line bundle. Moreover, α is a global meromorphic section of $L_\alpha \otimes K$ with a pole of order $r + 1$ at a singularity of rank r and, by assumption, no zeros. Therefore

$$L_\alpha \otimes K = - \sum (r_i + 1) \alpha_\alpha,$$

and so $\text{deg } L_\alpha = 2 - 2g - \sum (r_i + 1)$, where the sum is over the singularities and g is the genus.

Now a point $x \in X$ is the image of $(x, e) \in X \times G$ under the projection along F . So we have

$$N_x = T_{(x,e)}(X \times G) / (T_x \oplus L_\alpha).$$

Thus there is a short exact sequence

$$0 \rightarrow L_A \rightarrow \mathfrak{g} \rightarrow N \rightarrow 0,$$

where \mathfrak{g} is the trivial bundle \mathfrak{g} -bundle over X .

When $X = \mathbb{C}P_1$ and $\sum (r_i + 1) \geq 4$, we have $H^1(\mathfrak{g}) = 0$, $H^0(L_A) = 0$, and

$$\dim H^1(L_A) = -3 + \sum (r_i + 1).$$

From the corresponding long exact sequence, therefore, $H^1(N) = 0$, and

$$0 \rightarrow H^0(\mathfrak{g}) \rightarrow H^0(N) \rightarrow H^1(L_A) \rightarrow 0.$$

It follows that

$$d_1 = 0, \quad d_0 = \sum (r_i + 1) - 3 + \dim G.$$

We conclude that X is one of a d_0 -parameter of embedded copies of \mathbb{CP}_1 . If all the singularities are regular, then d_0 is the dimension of the space of configurations of poles ($\sum(r_i + 1) - 3$), plus the dimension of G . The deformations of X are parametrised by the positions of the poles, modulo the action of $SL(2, \mathbb{C})$ on \mathbb{CP}_1 , together with constant gauge transformations.

Example (The Schlesinger equations). Suppose that $X = \mathbb{CP}_1$ and that all the singularities are regular. We choose the domains of the Ω_i s so that each pole lies in only one. The Ω_i s are holomorphic and the deformation is determined by the holomorphic tangent vector fields T_{ij} on the overlaps of the domains. Since $H^1(X, TX) = 0$, we have $T_{ij} = T_i - T_j$, where T_i is holomorphic on the domain of Ω_i (but possibly non-zero at the corresponding pole). So if we put $Y = Y_i + T_i$, then Y is a global section of $T\mathcal{Z}|_X$.

Let z be a global stereographic coordinate on X , with $z = \infty$ not one of the poles, and use Y to transfer z along the deformations of X . Then

$$\alpha = - \sum \frac{A_i dz}{z - a_i},$$

where the a_i s are positions of the poles and the coefficients A_i are independent of z and satisfy

$$\sum_i A_i = 0$$

(since there is no pole at infinity). Put $\Omega = i_Y \theta$. Then Ω is meromorphic with simple poles at the a_i s and

$$\frac{\partial A}{\partial t} = \frac{\partial \Omega}{\partial z} - [A, \Omega].$$

In fact, since Ω_i is holomorphic at a_i and $T_i(a_i) = \partial a_i / \partial t$, we have

$$\Omega = - \frac{\partial a_i}{\partial t} \frac{A_i}{z - a_i} + O((z - a_i)^0).$$

It follows that

$$\Omega = - \sum_i \frac{\partial a_i}{\partial t} \frac{A_i}{z - a_i} + k,$$

where k is a matrix independent of z . Therefore

$$\frac{\partial A_i}{\partial t} = - \sum_{j \neq i} \frac{\partial a_j}{\partial t} \frac{[A_i, A_j]}{a_i - a_j} + [k, A_i],$$

which is a form of the Schlesinger equations. The last term is simply a infinitesimal gauge transformation; the first gives the dependence of the A_i s on the configuration of the poles.

4.2. *The full case*

In a full twistor space, d_0 is generally larger than in the minimal construction. Here we can find N in another way.

We suppose that all the singularities are generic. Then, by (5) and Lemma 1, a local section V of $T\mathcal{Z}|_X$ is a map Ω from a neighbourhood in X to \mathfrak{g} such that a singularity at of rank r ,

$$\Omega = O(z^{-r}), \quad [\alpha, \Omega] = O(z^{-r-1}),$$

where z is a local coordinate that vanishes at the singularity. That is, in a local gauge in which the singular part of α is diagonal, the diagonal entries in Ω have poles of at most order r , and the off-diagonal entries are holomorphic. Thus these algebraic conditions characterise Ω as a local section of a holomorphic bundle $E \rightarrow X$ with fibre \mathfrak{g} (and therefore rank $n^2 - 1$) and degree $\sum(n - 1)r_i$. In the full case, therefore, we have that $T\mathcal{Z}|_X = E$ can be constructed directly from the positions and ranks of the singularities of the ODE on X . Put

$$L = TX \otimes \sum(-a_i),$$

so that a local holomorphic section of L is a tangent vector field that vanishes at the poles. Then we have a short exact sequence

$$0 \rightarrow L \rightarrow E \rightarrow N \rightarrow 0$$

with the second map given by contraction with α . Hence there is an exact sequence

$$0 \rightarrow H^0(L) \rightarrow H^0(E) \rightarrow H^0(N) \rightarrow H^1(L) \rightarrow H^1(E) \rightarrow H^1(N) \rightarrow 0.$$

If the genus of X is g , and if there are k singularities in total, then

$$\deg(L) = 2 - 2g - k, \quad \dim H^0(E) - \dim H^1(E) = (n^2 - 1)(1 - g) + (n - 1) \sum r_i$$

(the latter identity coming from the Riemann–Roch theorem).

A global section of E is a meromorphic map $\Omega : X \rightarrow \mathfrak{g}$ such that at a singularity of rank r ,

$$\Omega = z^{-r} g_f q g_f^{-1} + O(z^0),$$

where z is a local coordinate that vanishes at the singularity and q is a diagonal polynomial of degree $r - 1$. When $X = \mathbb{C}P_1$, Ω is determined as a global rational map by the q s up to the addition of a constant element of \mathfrak{g} , and the q s can be specified independently. In this case, therefore, $\dim H^0(E) = (n - 1) \sum r_i + \dim G$, and $\dim H^1(E) = 0$. Moreover if $k \geq 4$, then

$$\dim H^0(L) = 0, \quad \dim H^1(L) = k - 3.$$

It follows that

$$d_0 = H^0(N) = (n - 1) \sum r_i + \dim G + k - 3, \quad d_1 = \dim H^1(N) = 0.$$

If either $n = 2$ ($G = \text{SL}(2, \mathbb{C})$) or $\sum r_i = 0$ (all singularities regular), then d_0 is the same as in the minimal case; in either of these cases, the minimal twistor space is full and, by Proposition 8 below, it gives all possible isomonodromic deformations. In general, however, there are more isomonodromic deformations than are given by the minimal construction: the additional parameters are the coefficients of the diagonal polynomials t (of degree $r - 1$) at the irregular singularities ($r \geq 1$).

When X has higher genus, $\dim H^1(E) = \dim H^0(E^* \otimes K)^*$ is generically zero, whenever $(n - 1) \sum r_i > n^2 g + g - 2$.

4.3. Twistor curves

Let $dy + \alpha y = 0$ be a generic system on a compact Riemann surface X and suppose that $H^1(E) = 0$. Then we can construct a full twistor space (\mathcal{Z}, X) . Since $H^1(X, N) = 0$, X is one of a complete holomorphic family \mathcal{K} of curves $X \subset \mathcal{Z}$.

Proposition 8. *Let (X_t, α_t) , $t \in [0, 1]$, be an isomonodromic deformation of (X, α) . Then for small t , there is a path X_t in \mathcal{K} such that $\alpha_t = \theta|_{X_t}$.*

Proof. Let y_t be solution to

$$dy_t + \theta_t y_t = 0$$

with constant monodromy, and with constant connection matrices C_i to the special solutions at the poles.

Let $z, z' \in X$ be nearby points (neither a pole) and let $g, g' \in G$ be close to the identity. Then, by integrating the action of g on \mathcal{Z} , we have two points $zg, z'g'$ in \mathcal{Z} near X . These are the same if

$$gg'^{-1} = y_0(z)y_0(z')^{-1}. \quad (15)$$

Let $z_t \in X_t$ vary continuously with t , and suppose that z_t is not a pole for any small t . Put

$$\rho_t(z_t) = z_0 y_0(z_0) y_t(z_t)^{-1} \in \mathcal{Z} \quad (16)$$

(the right-hand side is interpreted by regarding $z_0 \in X$ as a point of $X \subset \mathcal{Z}$ and by using the local action of G on \mathcal{Z}). This is independent of the choice of branch of y_t and y_0 (so long as we make the choice of branch continuously) since y_0 and y_t have the same monodromy. Moreover, $\rho_t(z_t)$ depends only on z_t , and not on the path, by (15). So if we exclude a small neighbourhood of each pole in X_t , then we can embed the complement in \mathcal{Z} by $z_t \mapsto \rho_t(z_t)$. By fixing z_0 and moving z_t , we see from (16) that $\rho_t^* \theta = \alpha_t$.

It remains to show that ρ_t extends holomorphically to the poles. Consider one of the poles (a point of X_t , varying continuously with t). We can choose a coordinate z in a neighbourhood D of the pole on each X_t so that D is the unit disc and the pole is at $z = 0$. Then, for small t , since \mathcal{Z} is full, there exists a holomorphic map $\gamma_t : D \rightarrow \mathcal{Z}$ such that $\alpha'_t = \gamma_t^* \theta$ has the same singularity data at $z = 0$ as α_t has at $z = 0$. Since α_t is an isomonodromic deformation, α_t and α'_t also have the same Stokes' matrices.

Let y'_t be a solution to

$$dy'_t + \alpha'_t = 0$$

with the same monodromy and connection matrices to the special solutions in the sectors at $z = 0$ as y_0 . Then

$$\gamma_t(z) = zy_0(z)y'_t(z)^{-1}.$$

Further $y'_t y_t^{-1}$ is holomorphic at $z = 0$. This is because it is single-valued, since y_t and y'_t have the same holonomy, and bounded since in any sector \mathcal{S} at $z = 0$

$$y'_t y_t^{-1} = y'_\mathcal{S} y_\mathcal{S}^{-1} \sim g'_f g_f^{-1},$$

where $y'_\mathcal{S}$, $y_\mathcal{S}$ are the corresponding special solutions and g'_f , g_f are the formal gauge transformations to diagonal form. So the embedding ρ_t extends by mapping $z \in D \subset X_t$ to $\gamma_t(z)y'_t y_t^{-1}$. \square

4.4. Isomonodromic flows for systems on the Riemann sphere

The number of independent isomonodromic deformations (the dimension of \mathcal{K}) of a generic system on $X = \mathbb{CP}_1$,

$$\dim H^0(X, N) = (n - 1) \sum r_i + \dim G + k - 3.$$

We shall now show that the deformations are given by Hamiltonian flows on symplectic manifolds constructed from the affine orbits in $L\mathfrak{g}^*$.

In this case, the twistor curves in \mathcal{Z} are copies of \mathbb{CP}_1 , and they can be parametrised by a global stereographic coordinate $z \in \mathbb{C} \cup \{\infty\}$. We denote by $\hat{\mathcal{K}}$ the space of parametrised curves, which has dimension

$$\dim \hat{\mathcal{K}} = (n - 1) \sum r_i + \dim G + k.$$

The points of $\hat{\mathcal{K}}$ can be labelled by the positions of the poles (k parameters), the polynomials t at each pole ($(n - 1) \sum r_i$ parameters) and a choice of gauge ($\dim G$ parameters).

An element of $\hat{\mathcal{K}}$ is a mapping $\rho : \mathbb{CP}_1 \rightarrow \mathcal{Z}$ from some fixed copy of the Riemann sphere. It determines a rational \mathfrak{g} -valued 1-form

$$\alpha = \rho^* \theta = -A dz,$$

where A is rational, with poles of order $r_i + 1$ at k points a_1, \dots, a_k (we assume that none of the poles is at infinity, so $A = O(z^{-2})$ as $z \rightarrow \infty$). In a neighbourhood of a_i , we put $z_i = z - a_i$ and assume, without loss of generality, that no other pole lies in the closure of the disc $D_i = \{|z_i| < 1\}$. Then $\alpha_i = \alpha|_{D_i}$ determines a point of the symplectic manifold \mathcal{M}_{r_i} . Thus we have a map

$$\hat{\mathcal{K}} \rightarrow \mathcal{M} = \mathcal{M}_{r_1} \times \mathcal{M}_{r_2} \times \dots \times \mathcal{M}_{r_k}.$$

It is not surjective, since a given point of \mathcal{M} is not, in general, given by the restrictions of a global 1-form α . However, α is uniquely determined by the positions of its poles and

by its image in \mathcal{M} , since the difference between two α s with the same pole positions, and determining the same point of \mathcal{M} , is a global holomorphic 1-form, and therefore vanishes.

Given $[\alpha_i] \in \mathcal{M}_{r_i}$ and the points $a_i \in \mathbb{CP}_1$, we put $\alpha_i = -A_i dz_i$ and denote by A_{i-} and A_{i+} the negative and non-negative degree terms in the Laurent expansion of A_i in powers of z_i in a neighbourhood of a_i . Given also a diagonal polynomial q_i of degree $r_i - 1$, we put

$$\Omega_{q_i} = (z^{-r_i} g_f^{-1} q_i g_f)_-,$$

where g_f is the formal gauge transformation to the diagonal form of α_i and again the minus subscript denotes the negative terms in the Laurent expansion in powers of z_i . Then A_{i-} and Ω_{q_i} are global meromorphic functions on \mathbb{CP}_1 with values in \mathfrak{g} ; they are holomorphic except at a_i , where they have poles of order $r_i + 1$ and r_i , respectively. Moreover, A_{i-} and Ω_{q_i} are independent of the choice of representative in $[\alpha_i]$.

Proposition 9. *The isomonodromic deformations of a generic $SL(n, \mathbb{C})$ system on \mathbb{CP}_1 are generated by the Hamiltonians*

$$h_v = \sum_j \frac{1}{2\pi i} \oint \text{tr}(\alpha_j v), \quad h_i = \sum_{j \neq i} \frac{1}{2\pi i} \oint \text{tr}(\alpha_j A_{i-}), \quad h_{q_i} = \sum_j \frac{1}{2\pi i} \oint \text{tr}(\alpha_j \Omega_{q_i})$$

on \mathcal{M} , where v is a constant element of \mathfrak{g} and the integrals are around small circles surrounding the poles.

The h_i 's are time-dependent Hamiltonians, the 'times' being the positions a_i of the poles.

Proof. First we note that the Hamiltonians h_v generate the constant gauge transformations. Consider next the flow generated by h_i . We shall find the value of the Hamiltonian vector field at a point of \mathcal{M} constructed from a global meromorphic 1-form α . To do this, we must evaluate the gradient of h_i at such a point. We have

$$\delta h_i = \sum_{j \neq i} \frac{1}{2\pi i} \oint \text{tr}(\delta \alpha_j A_{i-} + \alpha \delta A_{i-}) = -\frac{1}{2\pi i} \oint \text{tr}(\alpha_i \delta A_{i-}) + \sum_{j \neq i} \frac{1}{2\pi i} \oint \text{tr}(\delta \alpha_j A_{i-}).$$

However

$$\frac{1}{2\pi i} \oint \text{tr}(\alpha_i \delta A_{i-}) = \frac{1}{2\pi i} \oint \text{tr}(\alpha_{i+} \delta A_i) = \frac{1}{2\pi i} \oint \text{tr}(A_{i+} \delta \alpha_i).$$

We conclude that the value of the Hamiltonian vector field at such a point is

$$\delta \alpha_i = -\nabla A_{i+}, \quad \delta \alpha_j = \nabla A_{i-}, \quad j \neq i.$$

The claim is that this is tangent to an isomonodromic deformation. To see this, let y be a solution to $dy + \alpha y = 0$, let D be a disc containing a_i , but no other pole, and let D' be a second disc not containing a_i such that D, D' is an open cover of \mathbb{CP}_1 . For small t , put $F_t(z) = y(z-t)y(z)^{-1}$. Then $F : D \cap D' \rightarrow G$ is single-valued, holomorphic, and equal

to the identity when $t = 0$. By Birkhoff’s theorem, $F_t = h_t^{-1}h'_t$ for some holomorphic maps $h_t : D \rightarrow G, h'_t : D' \rightarrow G$, with $h'_t(\infty) = 1$. Put

$$y_t(z) = \begin{cases} h_t(z)y(z-t), & z \in D, \\ h'_t(z)y(t), & z \in D', \end{cases}$$

and $\alpha_t = -dy_t y_t^{-1}$. Then the definitions agree on $D \cap D'$ and α_t is a global meromorphic 1-form with poles at $z = a_j (j \neq i)$ and $z = a_i + t$. Moreover, $\partial_t y_t = \Omega_t y_t$, where

$$\Omega_t = \begin{cases} \partial_t h_t h_t^{-1} - h_t A_i(z-t) h_t^{-1} & \text{in } D, \\ \partial_t h'_t h'^{-1}_t & \text{in } D'. \end{cases}$$

Since h_t and h'_t are holomorphic in D and D' , it follows that the deformation is isomonodromic (see Proposition 11).

At $t = 0$, we have $h_t = h'_t = 1$ and

$$\partial_t h'_t = A_{i-}, \quad \partial_t h_t = -A_{i+},$$

we also have at $t = 0$ that $\delta\alpha_i = \nabla(\partial_t h_t), \delta\alpha_j = \nabla(\partial_t h'_t)$ for $j \neq i$. So the tangent to the deformation is the Hamiltonian vector field constructed above: these deformations move the poles, but leave the singularity data unchanged.

Now consider the flow generated by h_{q_i} . Proceeding as before to calculate the value of the Hamiltonian vector field at a point given by a global 1-form α , we have

$$\delta h_{q_i} = \sum_j \frac{1}{2\pi i} \oint \text{tr}(\delta\alpha_j \Omega_{q_i} + \alpha_j \delta\Omega_{q_i}) = \sum_j \frac{1}{2\pi i} \oint \text{tr}(\alpha_j \delta\Omega_{q_i}).$$

So in this case, the value of the Hamiltonian vector field is

$$\delta\alpha_i = \nabla_\alpha \Omega_{q_i},$$

which is clearly isomonodromic. These deformations change the singularity data at a_i , leaving the position of the poles unchanged. □

Remark. The Hamiltonians H_{q_i} vanish on the orbit of a global 1-form α , while

$$h_i = \frac{1}{\tau} \frac{\partial \tau}{\partial a_i},$$

where τ is as in [9].

Example. In the case of a generic system with only regular singularities, $\mathcal{M} = \mathcal{M}_1 \times \dots \times \mathcal{M}_1$. The tangent space is spanned \mathcal{M}_1 is spanned by the generators of constant gauge transformations. If we write

$$\alpha = -A dz = \sum \frac{A_i dz}{z - a_i}, \quad \alpha_i = -\frac{A_i dz_i}{z_i},$$

then we can identify \mathcal{M} (as a symplectic manifold) with the product of the coadjoint orbits of the A_i in \mathfrak{g}^* . In this case,

$$A_{i-} = \frac{A_i}{z - a_i},$$

there are no h_{q_i} s, while

$$h_i = \sum_{j \neq i} \frac{\text{tr}(A_i A_j)}{a_i - a_j}.$$

Acknowledgements

This paper draws heavily on ideas on Nigel Hitchin as well as on Philip Boalch's D.Phil. Thesis (on which [2] is based). I thank Majiec Dunajski and Guido Sanguinetti for many interesting discussions, and particularly Philip Boalch for his comments and for the corrections he made to the first draft.

Appendix A. Singularities of systems of ODEs

Let A be a meromorphic map from some neighbourhood D of the origin in \mathbb{C} into \mathfrak{g} (the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$) with a pole of order $r + 1$ at $z = 0$. Then the system

$$\frac{dy}{dz} = Ay, \tag{A.1}$$

has a *singularity of Poincaré rank r* at the origin. It is *regular* or *Fuchsian* if $r = 0$, and *irregular* if $r > 0$. In this paper, y will always be a *matrix fundamental solution* — i.e., it will take values in G .

A.1. Gauge and point transformations

When we regard the system as a connection on a vector bundle, we must allow *gauge transformations* (changes of trivialisation) of the form

$$A \mapsto g^{-1}Ag - g^{-1} \frac{dg}{dz}, \quad \nabla \mapsto g^{-1} \circ \nabla \circ g, \quad y \mapsto gy,$$

where $g : X \rightarrow G$ is holomorphic. When g is constant, A transforms by conjugation. We also admit transformations of the coordinate $z \mapsto \hat{z}$, under which

$$A \mapsto \hat{A} = A \frac{dz}{d\hat{z}}.$$

If this is to fix the singularity at the origin, then we must take $\hat{z}(0) = 0$.

A.2. Generic irregular singularities

An irregular singularity of rank r is *generic* if the eigenvalues of

$$A_{-r-1} = z^{r+1}A|_{z=0}$$

are distinct. In this case, we can assume, without loss of generality, that the eigenvalues of $z^{r+1}A$ are distinct throughout the neighbourhood. If we choose an ordering for the eigenvalues, then we can find a holomorphic map $g : D \rightarrow G$ such that $z^{r+1}g(z)A(z)g^{-1}(z)$ is holomorphic and diagonal, with the eigenvalues as diagonal entries.

It follows that we can find a holomorphic map $D \rightarrow G$ and a diagonal polynomial $p(z)$ of degree r such that

$$A - g(z)^{-1} \frac{p(z)}{z^{r+1}} g(z)$$

is holomorphic at the origin. Therefore we have the normal form:

$$A \sim d \left(\frac{t}{z^r} \right) + \frac{m dz}{z} + R(z), \tag{A.2}$$

where ‘ \sim ’ denotes gauge equivalence, t is a diagonal polynomial of degree $r - 1$, m is a constant diagonal matrix called the *exponent of formal monodromy*, and the remainder R is holomorphic at $z = 0$. Given the local coordinate z and the ordering of the eigenvalues, t and m are uniquely determined by A , independently of the choice of gauge. We call them *the singularity data* at 0. If the ordering is changed, then the diagonal entries are permuted; if the coordinate is changed, then m is unchanged, while $t \mapsto t'$, where t' is obtained from t by making the coordinate transformation and truncating the Taylor series in z .

If one looks for a gauge transformation $g(z) = g_0 + g_1z + \dots$ such that (A.2) holds with $R = 0$, then the coefficients g_i can be determined uniquely once a choice has been made for g_0 to diagonalise A_{-r-1} . For each such choice, one can, therefore, find a unique formal solution

$$y_f = g_f(z) \exp(Tz^{-r} + m \log z).$$

In general, the formal series does not converge. However, by truncating, one can make R vanish to arbitrarily high order at $z = 0$.

A.3. Sectors and special solutions

The eigenvalues λ_i of A_{-r-1} determine a sequence of *Stokes’ rays* through the origin, on which $\text{Re}(z^{-r}(\lambda_i - \lambda_j))$ changes sign for some pair of eigenvalues. Given a pair of consecutive rays (in order around the unit circle), with arguments θ_1, θ_2 , we define a *sector* \mathcal{S} by

$$\mathcal{S} = \left\{ z \mid \theta_1 - \frac{\pi}{2r} < \arg(z) < \theta_2 + \frac{\pi}{2r} \right\}. \tag{A.3}$$

For each such sector \mathcal{S} , there is a unique *special solution*

$$y_{\mathcal{S}} = g_{\mathcal{S}}(z) \exp(tz^{-r} + m \log z),$$

such that $g_{\mathcal{S}} \sim g_f$ as $z \rightarrow 0$ in \mathcal{S} . (See, e.g. [2] for a careful account of this proposition).

The solutions $y_{\mathcal{S}}$ are independent of the choice of coordinate (as maps $D \rightarrow G$). They are uniquely determined in each sector by the choice of g_0 . So, as sections of the principal bundle over each sector, they are determined uniquely by the choice of the frame at the origin in which A_{-r-1} is diagonal.

A.4. Regular singularities

In the regular case ($r = 0$), (A.2) still holds, with $t = 0$, provided that the eigenvalues of A_{-1} are distinct. The formal series for g can be found provided that, in addition, no two eigenvalues differ by an integer. It then necessarily converges, so y_f is a solution.

A.5. Global systems

Let X be a compact Riemann surface. For each k -tuple $\mathbf{r} = (r_1, \dots, r_k)$ of non-negative integers, we denote by $\mathcal{D}_{\mathbf{r}}(X, E)$, or simply by $\mathcal{D}_{\mathbf{r}}$, the space of meromorphic $\mathfrak{sl}(n, \mathbb{C})$ -connections $\nabla = d + \alpha$ on the trivial vector bundle $E = X \times \mathbb{C}^n$ with k poles of order (at most) $r_1 + 1, \dots, r_k + 1$.⁸

If we choose a local trivialisation and a coordinate z that vanishes at one of the poles, a connection $\nabla \in \mathcal{D}_{\mathbf{r}}$ is given in a neighbourhood of the pole by (A.1).⁹

A.6. The monodromy representation

A local G -valued solution y to the equation $d + \alpha y = 0$ can be continued analytically: it is singular at the poles, and multi-valued (single-valued on the covering space of the complement of the poles). If

$$\gamma : [0, 1] \rightarrow \mathbb{C}\mathbb{P}_1 \setminus \{a_1, \dots, a_k\}, \quad \gamma(0) = \gamma(1) = z_0$$

is a closed loop and z_0 is some fixed base point, then we have $y(\gamma(1)) = M_{\gamma} \in G$ for some *monodromy matrix* M_{γ} which depends only on the homotopy class of γ .

Definition 3. The monodromy representation of (2) it is the homomorphism

$$\pi_1(\mathbb{C}\mathbb{P}_1 \setminus \{a_1, \dots, a_k\}) \rightarrow G : [\gamma] \mapsto M_{\gamma}.$$

The monodromy representation is independent of z_0 and the choice of y up to conjugation by a fixed element of G .

⁸ Note that α determines a holomorphic map $X \rightarrow \mathbb{P}\mathfrak{g}$.

⁹ When $X = \mathbb{C}\mathbb{P}_1$, we have $\alpha = -A dz$, where z is a stereographic coordinate and A is defined globally as a rational section of $\mathfrak{g} \otimes \mathcal{O}(-2)$. If all its singularities are at finite values of z , then A has N poles and $A = \mathcal{O}(z^{-2})$ as $z \rightarrow \infty$; but if one of the singularities is at infinity, then $A = \mathcal{O}(z^{r-1})$ as $z \rightarrow \infty$, where r is the rank.

A.7. Deformations

Definition 4. Let $\nabla_0, \nabla_1 \in \mathcal{D}_r(X)$. A deformation of ∇_0 into ∇_1 is a smooth path $\nabla_t \in \mathcal{D}_r(X)$, $t \in [0, 1]$, from ∇_0 to ∇_1 .

We are interested in the deformations of a connection ∇_0 into a second ∇_1 (or equivalently of α_0 into α_1) while preserving certain properties of the corresponding linear system.

Proposition 10. A deformation ∇_t has constant monodromy representation (up to conjugation) if and only if

$$\frac{d\nabla_t}{dt} = \nabla_t \Omega_t$$

for some family of holomorphic maps Ω_t (depending smoothly on t) from the complement of the poles of ∇_t in X into \mathfrak{g} .

There is an awkwardness in the terminology here: it is important to keep in mind that ‘having the same monodromy representation’ is not the same as ‘isomonodromic’ when irregular singularities are present.

Proof. By fixing a base point (disjoint from the poles) and a frame at the base point, we can find a solution y_t for each t which depends smoothly on t . If the monodromy representation is constant, then we can find a matrix $K_t \in G$ for each t such that the monodromy matrices of $y_t K_t$ are constant. If we take t close to t' and exclude small discs around the poles of ∇_t , then $g_{tt'} = y_t K_t K_{t'}^{-1} y_{t'}^{-1}$ is single-valued, and we can construct Ω_t in (C1) by putting

$$\Omega_t = \left. \frac{\partial g_{tt'}}{\partial t'} \right|_{t'=t}.$$

(This is holomorphic except at the poles of ∇_t .)

Conversely, if we are given Ω_t , then $d - A dz - \Omega dt$ is a flat connection on the trivial bundle over

$$(X \setminus \text{poles}) \times [0, 1].$$

Its holonomy coincides with the monodromy A_t for each t , and so the monodromy representation must be constant up conjugacy. \square

A.8. Isomonodromic deformations

We now consider deformations (X_t, ∇_t) in which we change both X and the connection ∇ .

Let a be a pole of ∇ and \mathcal{S} a sector at a . Then we have a solution $y_{\mathcal{S},a}$ to the system $\nabla y = 0$, uniquely determined up to the choice of the frame at a in which the leading coefficient of A is diagonal. If a' and \mathcal{S}' are another pole and sector at a' , and if γ is a path

with initial point near a in \mathcal{S} and endpoint near a' in \mathcal{S}' , then we can continue $y_{\mathcal{S},a}$ along γ . We shall have

$$y_{\mathcal{S},a} = y_{\mathcal{S}',a'} C_{\mathcal{S},\mathcal{S}',a,a',\gamma},$$

where $C_{\mathcal{S},\mathcal{S}',a,a',\gamma}$ is a constant matrix. These matrices are uniquely determined by A up to

$$C_{\mathcal{S},\mathcal{S}',a,a',\gamma} \mapsto D_a C_{\mathcal{S},\mathcal{S}',a,a',\gamma} D_{a'}^{-1},$$

where, for each pole a , D_a is the product of a diagonal matrix and a permutation matrix. The matrices connecting the special solutions in adjacent sectors at the same pole are called Stokes' matrices.

As we deform ∇ and X , we can vary \mathcal{S} , \mathcal{S}' , γ , and the special solutions continuously.

Definition 5. A deformation is isomonodromic if the exponents of formal monodromy and the matrices $C_{\mathcal{S},\mathcal{S}',a,a',\gamma}$ are constant, for an appropriate choice of special solutions.

An isomonodromic deformation has constant monodromy representation, but the converse is not true except in the Fuchsian case (all singularities regular). The following characterisation of the isomonodromy property is implicit in [9].

We can cover X_t by discs D_i varying continuously with t such that each pole lies in just one disc. On each disc D_i , we can choose a coordinate z_i such that, if $a_i \in D$ is a pole, then $z_i = 0$ at a_i , independently of t . We shall use these coordinates to identify the discs as t varies.

Proposition 11. A deformation (X_t, ∇_t) with constant monodromy representation is isomonodromic if and only if

1. in D_i , $d\nabla_t/dt = \nabla_t \Omega_{it}$ for some meromorphic $\Omega_{it} : D_i \rightarrow \mathfrak{g}$ (depending smoothly on t); Ω_{it} is holomorphic except possibly at a singularity of α_t , where it has a pole of order at most r_i if α_t has a singularity of rank r_i in D_i ;
2. on the intersection $D_i \cap D_j$ of two discs $\Omega_{it} - \Omega_{jt} = i_{T_{ij}} \alpha_t$ for some holomorphic vector field T_{ij} .

Proof. We shall look at the proof in outline. Suppose that the deformation is isomonodromic. Let y_t be a solution to $\nabla_t y_t = 0$, depending continuously on t and with constant monodromy (we have to keep in mind that y_t is multi-valued and singular at the poles).

Let D_i be a disc containing a pole (at $z_i = 0$) and put

$$g_{it'}(z_i) = y_{\mathcal{S}}(z_i) y_{\mathcal{S}'}^{-1}(z_i),$$

where $y_{\mathcal{S}}$ and $y_{\mathcal{S}'}$ are the special solutions at t and t' in the corresponding sectors at one of the poles. Then, for t' close to t , $g_{it'}$ is a single-valued holomorphic map $D_i \setminus \{0\} \rightarrow \mathfrak{g}$; it is independent of sector, because the Stokes' matrices are the same at t and t' . Once it is established that it is possible to differentiate the asymptotic expansions term-by-term, it is immediate that $\Omega_{it} = \partial_{t'} g_{it'}|_{t'=y}$ is meromorphic, and of the required form.

On a disc D_i that does not contain a singularity, we put $g_{it'} = y_t(z_i)y_{t'}(z_i)^{-1}$, and define Ω_{it} in the same way. If we choose the branch of y_t to vary continuously with t , $g_{it'}$ is independent of the choice of branch because the monodromy of y_t is independent of t .

Given y_t , the only freedom in the construction of Ω_{it} is in the choice of the coordinate z_i , and hence in the local identification of the discs on the different Riemann surfaces. A different choice for each t will add $i_T\alpha$ to Ω_{it} for some local holomorphic vector field T . Thus condition 2 holds on the overlap of two discs.

To prove the converse, suppose that Ω_{it} is meromorphic, as stated. Choose a continuously varying sector \mathcal{S} at the pole, and let

$$y_{\mathcal{S}}(z_i, t) = g_{\mathcal{S},a}(z_i) \exp(tz_i^{-r} + m \log z_i)$$

be the corresponding special solution. Then, by writing $\nabla_t = d_z - A_t dz_i$ and dropping the subscripts, we have

$$\frac{\partial}{\partial z} \left(\frac{\partial y}{\partial t} - \Omega y \right) = \frac{\partial A}{\partial t} y + A \frac{\partial y}{\partial t} - \frac{\partial \Omega}{\partial t} y - \Omega A y = A \left(\frac{\partial y}{\partial t} - \Omega y \right).$$

It follows that

$$\frac{\partial y}{\partial t} - \Omega y = yK$$

for some matrix K , which can depend of t but not z . Therefore

$$g_{\mathcal{S}}^{-1} \left[\frac{\partial g_{\mathcal{S}}}{\partial t} + g_{\mathcal{S}} \frac{\partial}{\partial t} \left(\frac{t}{z^r} \right) - \Omega g_{\mathcal{S}} \right] = e^{tz^{-r} + m \log z} K e^{-tz^{-r} - m \log z}.$$

The left-hand side is asymptotic to a power series, divided by z^{r+1} , as $z \rightarrow 0$ in \mathcal{S} (the same series for each sector at the pole). In the case $r > 0$, each off-diagonal entry on the right-hand side has an exponential factor which must blow up as $z \rightarrow 0$ along some directions in \mathcal{S} since the angle of the sector \mathcal{S} is more than π/r . This is a contradiction unless the off-diagonal entries in K all vanish. Thus K is a z -independent diagonal matrix. It can be absorbed into the special solutions to give that

$$\frac{\partial y_{\mathcal{S}}}{\partial t} = \Omega_t y_{\mathcal{S}},$$

and hence that the C matrices are constant. This is also true, more simply, in the regular case since the formal solutions then converge. \square

Appendix B. Symplectic form on \mathcal{C}_r

We prove here that the tensor in (14) is a symplectic form on \mathcal{C}_r .

Proposition 12. ω is a symplectic form on \mathcal{C}_r , independent of the choice of C_1 .

Proof. From the definitions, $C_{2r+1} = e^{-2\pi im} C_1 M$ and, in variational notation,

$$\delta S_i S_i^{-1} = C_i (C_i^{-1} \delta C_i - C_{i+1}^{-1} \delta C_{i+1}) C_i^{-1}. \tag{B.1}$$

We must show that ω is skew-symmetric, closed, and non-degenerate. From the first constraint, we have $\text{tr}(\delta S_i S_i^{-1} \delta' S_i S_i^{-1}) = 0$. It follows that

$$0 = \sum_1^{2r} \text{tr}(\delta S_i S_i^{-1} \delta' S_i S_i^{-1}) = \sum_1^{2r} \text{tr}((C_i^{-1} \delta C_i - C_{i+1}^{-1} \delta C_{i+1})(C_i^{-1} \delta' C_i - C_{i+1}^{-1} \delta' C_{i+1})).$$

However

$$\begin{aligned} & \sum_1^{2r} \text{tr}(C_{i+1}^{-1} \delta C_{i+1} C_{i+1}^{-1} \delta' C_{i+1}) \\ &= \sum_1^{2r} \text{tr}(C_i^{-1} \delta C_i C_i^{-1} \delta' C_i) - 4\pi^2 \text{tr}(\delta m \delta' m) - 2\pi i \text{tr}(\delta m \delta' C_1 C_1^{-1} + \delta' m \delta C_1 C_1^{-1}). \end{aligned}$$

The skew-symmetry follows. A similar calculation, starting from

$$\text{tr}(\delta S_i S_i^{-1} \delta' S_i S_i^{-1} \delta'' S_i S_i^{-1}) = 0,$$

shows that ω is closed. To show that ω is non-degenerate, we note that if $\omega(Y, \cdot) =$, then $\delta C_1 C_1^{-1} - \pi i \delta m$ is anti-diagonal, $P_i \delta C_i C_i P_i^{-1}$ is lower triangular for each i , and $P_i \delta C_i C_i^{-1} P_i^{-1}$ is upper triangular. However, from (B.1),

$$\delta C_{i+1} C_{i+1}^{-1} = S_i^{-1} \delta C_i C_i^{-1} S_i - S_i^{-1} \delta C_i C_i^{-1} S_i.$$

Therefore, $\delta C_i C_i^{-1}$ is diagonal and so $\delta C_i C_i^{-1} = \pi i \delta m$ for each i . It then follows from the second constraint (C2) in the definition of \mathcal{C}_r that $\delta m = 0$.

If we make a different choice for C_1 at each point, then the effect is to replace C_i by $C_i K$, where K is independent of i . This adds

$$\frac{1}{2\pi i} \sum_1^{2r} \text{tr}(C_i \delta K C_i^{-1} \delta' S_i S_i^{-1}) - \text{tr}(C_1 \delta K C_1^{-1} \delta' m)$$

to $\omega(Y, Y')$, which vanishes by (B.1). \square

Proof of Proposition 5. The manipulations are slightly more transparent in the classical variational notation, although it is straightforward to translate this into the language of differential forms.

We shall evaluate $\sigma(Y, Y')$ in (11) by putting $\Omega = \delta y y^{-1}$, $\Omega' = \delta' y y^{-1}$. We shall then shrink the contour to the origin and use the asymptotic behaviour of the y_i s.

For each i , choose $z_i \in \mathcal{S}_i \cap \mathcal{S}_{i+1}$ on the contour with $z_{2r+1} = z_1$, and define $\log z$ by making a cut along the ray through the origin and z_1 . On each sector,

$$\Omega = \Omega_i + y_i \delta C_i C_i^{-1} y_i^{-1},$$

where $\Omega_i = \delta y_i y_i^{-1}$. Moreover,

$$\nabla(y_i \delta C_i C_i^{-1} y_i^{-1}) = 0,$$

since C_i is independent of z . Therefore, in the notation of (11),

$$\frac{1}{2\pi i} \oint \text{tr}(\Omega \nabla \Omega') = \frac{1}{2\pi i} \sum_1^{2r} \int_{z_i}^{z_{i+1}} \text{tr}(\Omega_i \nabla \Omega'_i) + \frac{1}{2\pi i} \sum_1^{2r} (x_i(z_{i+1}) - x_i(z_i)), \quad (\text{B.2})$$

where z_i is some point on the contour in $\mathcal{S}_i \cap \mathcal{S}_{i+1}$, $x_i = \text{tr}(\delta C_i C_i^{-1} y_i^{-1} \delta' y_i)$, and the integrals are along segments of the contour. However,

$$x_{i+1} - x_i = \text{tr}(\delta C_i \delta C_i^{-1} \delta' S_i S_i^{-1} - S_i^{-1} \delta S_i y_{i+1}^{-1} \delta' y_{i+1}). \quad (\text{B.3})$$

This follows from the two relations $y_i = y_{i+1} S_i^{-1}$ and $C_i = S_i C_{i+1}$, which imply that

$$\text{tr}(\delta C_i C_i^{-1} y_i^{-1} \delta' y_i) = \text{tr}(\delta C_i C_i^{-1} S_i y_{i+1}^{-1} (\delta' y_{i+1}) S_i^{-1} - \delta C_i C_i^{-1} \delta' S_i S_i^{-1}),$$

and

$$S_i^{-1} \delta C_i C_i^{-1} S_i = \delta C_{i+1} C_{i+1}^{-1} - S_i^{-1} \delta S_i.$$

As $z \rightarrow 0$ in $\mathcal{S}_i \cap \mathcal{S}_{i+1}$, the second term on the right-hand side of (B.3) goes to zero by (13). Moreover,

$$x_{2r+1} - x_1 = 4\pi^2 \text{tr}(\delta m \delta' m) + 2\pi i \text{tr}(\delta C_1 C_1^{-1} \delta' m) - 2\pi i (\delta m y_1^{-1} \delta y_1).$$

To deal with the first term in (B.2), we note that

$$\Omega_i = \Theta + g \delta m g^{-1} \log z + O(z^N)$$

as $z \rightarrow 0$ in \mathcal{S}_i for some large N (depending on the truncation of the formal power series). Therefore in \mathcal{S}_i

$$\begin{aligned} \text{tr}(\Omega_i \nabla \Omega'_i) &= \text{tr}(\Theta \nabla \Theta') + \text{tr} \left[\left(g^{-1} \delta g + \frac{\delta t}{z^r} \right) \delta' m - \left(g^{-1} \delta' g + \frac{\delta' t}{z^r} \right) \delta m \right] \frac{dz}{z} \\ &\quad + \partial_z (\text{tr}(\delta' m (g^{-1} \delta g + z^{-r} \delta t)) \log z) dz + \text{tr}(\delta m \delta' m) \log z \frac{dz}{z} + O(N'), \end{aligned}$$

for some large N' . We therefore have

$$\begin{aligned} \frac{1}{2\pi i} \sum_1^{2r} \int_{z_i}^{z_{i+1}} \text{tr}(\Omega_i \nabla \Omega'_i) &= \frac{1}{2\pi i} \oint \text{tr}(\Theta \nabla \Theta') + \text{tr}(\gamma \delta' m - \gamma' \delta m) + \text{tr}(\delta' m y_1^{-1} \delta y_1)_{z_1} \\ &\quad + \pi i \text{tr}(\delta m \delta' m) + \epsilon \end{aligned}$$

where $\epsilon \rightarrow 0$ as the contour is shrunk. The proposition follows by putting the two terms together, by shrinking the contour towards $z = 0$, and by using the definition (14). \square

References

- [1] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, LMS Lecture Notes, Vol. 149, Cambridge University Press, Cambridge, 1991.

- [2] P. Boalch, Symplectic manifolds and isomonodromic deformations, Preprint, SISSA, Trieste, 2000.
- [3] B. Dubrovin, Painlevé transcendents in two-dimensional topological field theory, in: R. Conte (Ed.), *The Painlevé Property, One Century Later*, CRM Series in Mathematics, Springer, New York, 1999.
- [4] H.M. Farkas, I. Kra, *Riemann Surfaces*, Springer, New York, 1991.
- [5] N.J. Hitchin, Twistor spaces, Einstein metrics and isomonodromic deformations, *J. Diff. Geom.* 42 (1995) 30–112.
- [6] N.J. Hitchin, Poncelet polygons and the Painlevé equations, in: *Geometry and Analysis*, Bombay, 1992, Tata Institute, Bombay, 1995, pp. 151–185.
- [7] N.J. Hitchin, A new family of Einstein metrics, in: *Manifolds and Geometry*, Sympos. Math., Vol. XXXVI, Pisa, 1993, Cambridge University Press, Cambridge, 1996, pp. 190–222.
- [8] N.J. Hitchin, Geometrical aspects of Schlesinger’s equation, *J. Geom. Phys.* 23 (1997) 287–300.
- [9] M. Jimbo, T. Miwa, K. Ueno, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients. I. General theory and τ -functions, *Physica D* 2 (1981) 306–352.
- [10] M. Jimbo, T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II, *Physica D* 2 (1981) 407–448.
- [11] M. Jimbo, T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients III, *Physica D* 4 (1981) 26–46.
- [12] K. Kodaira, A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, *Ann. Math.* 75 (1961) 146–162.
- [13] B. Malgrange, Sur les déformations isomonodromiques, *Séminaire de l’Ecole Norm. Sup. IV* (1982) 401–426.
- [14] L.J. Mason, N.M.J. Woodhouse, *Integrability, Self-duality, and Twistor Theory*, Oxford University Press, Oxford, 1996.
- [15] A. Pressley, G. Segal, *Loop Groups*, Oxford University Press, Oxford, 1986.
- [16] M. Sato, T. Miwa, M. Jimbo, Holonomic quantum fields: I, *Publ. RIMS, Kyoto* 14 (1978) 223–267.
- [17] M. Sato, T. Miwa, M. Jimbo, Holonomic quantum fields: II–IV, *Publ. RIMS, Kyoto* 15 (1979) 201–267, 577–629, 871–972.
- [18] M. Sato, T. Miwa, M. Jimbo, Holonomic quantum fields: V, *Publ. RIMS, Kyoto* 16 (1980) 531–584.
- [19] J.-M. Souriau, *Structures des Systèmes Dynamique*, Dunod, Paris, 1970.
- [20] N.M.J. Woodhouse, *Geometric Quantization*, 2nd Edition, Oxford University Press, Oxford, 1992.